Note To File
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Date: 140927
Revised: 141208
Revised: 141221
Revised: 150101

Atwood's Machine

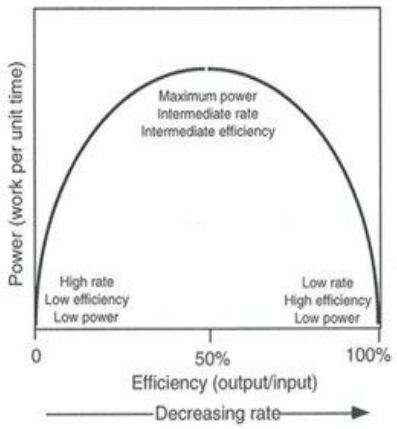
1 References

1. Wikipedia, "Atwood Machine", http://en.wikipedia.org/wiki/Atwood_Machine
2. "Robert Costanza, "Maximum Power Principle", http://www.eoearth.org/view/article/154526/
3. CAS Hall, "The continuing importance of Maximum Power", Ecological Modelling, 178 (2004) 107-113.
4. 140927 Atwood's Machine R4.xls, my spreadsheet.
5. 140121 ODD Description of EiLab, Model I, R5.pdf
6. 140124 PPR - Definition of EI with equations R12.docx
7. 130919 PPR - The Enveloping Curve R4.docx

2 Purpose

To understand the physics of Atwood's Machine and its connection to the strange curve associated with it.

Figure 1 – From Ref 2.



3 Description

In Atwood's machine two masses are hung over a pulley. One mass is a high above the floor, and the other rests on the floor. The higher mass is heavier. As gravity pulls the higher mass down, the lower mass rises. The goal is to use the heavier mass to do work by raising the lower mass. Thus the 'useful work' is only the work performed in raising the lower mass.

The goal here is to replicate the graph shown in Ref 2 and to understand how it comes together. (Shown at right.)

3.1 Assignment of Variable Names

In this NTF I use a number of variables, as follows:

- M_h = higher heavier mass; the h stands for heavier, as the masses switch places later.
- M_l = lower lighter mass; the l stands for lighter.

- D = distance from M_h to floor at start, i.e. the distance that M_h falls; D is also the distance that M_l rises because they are coupled.
- T = time required for M_h to fall to the floor. T is also the time it takes for M_l to rise because they are coupled.
- $M_t = M_h + M_l$ = the sum of the masses.
- $M_d = M_h - M_l$ = the difference between the masses.
- g = acceleration due to gravity ≈ 9.8 m/s^2.
- A = actual acceleration of the coupled mass assembly (M_l rising and M_h falling).
- F_h and F_l = gravitational forces on M_h and M_l respectively.
- F_t = total force on coupled mass assembly.
- Other variables will be defined as needed.

4 Step A – Solve for actual acceleration (A) of the coupled mass assembly

$F_t = F_h - F_l$
But $F_t = M_t * A$
So $(F_h - F_l) = M_t * A$
$(M_h g - M_l g) = M_t * A$
$(M_h - M_l) * g = M_t * A$

$A = g * \frac{M_d}{M_t}$	Equ 1

5 Step B – Solve for the drop time – T_d

$D = ½ * A * T^2$
$T_d = (2*D/A)^{1/2}$

$T_d = \left(\frac{2D}{A}\right)^{1/2} = \left(\frac{2DM_t}{gM_d}\right)^{1/2}$	Equ 2

6 Step C – Solve for the velocity of the mass assembly when the heavy mass hits the floor.

V_f = final velocity = AT_d

$V_f = AT_d = (2DA)^{1/2} = \left(\frac{2DgM_d}{M_t}\right)^{1/2}$	Equ 3

7 Step D – Solve for the total kinetic energy of the system when the heavy mass hits the floor.

E_k = kinetic energy at end = ½ * M_t * V_f^2

$E_k = \left(\frac{M_t}{2}\right)\left(V_f{}^2\right) = \frac{M_t}{2} \times \frac{2DgM_d}{M_t} = DgM_d$	Equ 4

This kinetic energy becomes waste heat when the heavier mass M_h (now the lower mass) hits the floor (transferring its kinetic energy as heat to the floor). [What about the kinetic energy of the mass M_l? It is expended as sound and heat as it bobs about at the top of its climb and the elasticity of the string at first absorbs energy, then expends it.]

We could also call this the work expended (W_e) as the Atwood's Machine runs.

W_e is equivalent by definition to E_k

8 Step E – Make formulae for:

a) The useful work done – W_u is calculated as force times distance:

$W_u = DgM_l$	Equ 5

b) The total work done – W_t is calculated as the sum of wasted energy and useful work:

$W_t = E_k + W_u$

$W_t = DgM_d + DgM_l = DG(M_h - M_l + M_l) = DgM_h$	Equ 6

c) The efficiency – E is calculated as the useful work over total work:

$E = W_u/W_t$

$\mathrm{E} = \frac{DgM_l}{DgM_h} = \frac{M_l}{M_h}$	Equ 7

d) The Power – P_s is calculated as useful work per second.

$P_s = W_u/T$

$\mathrm{P}_s = \frac{DgM_l}{\left(\frac{2DM_t}{gM_d}\right)^{1/2}} = \left(\frac{(DgM_l)^2 gM_d}{2DM_t}\right)^{1/2} = M_l\left(\frac{Dg^3M_d}{2M_t}\right)^{1/2}$	Equ 8

D, g and Mt are constants. M_l and M_d are variables.

I note that this concept of 'power' is really tricky and I am unsure that I understand it. For example, if we are talking about a light bulb, the power of the bulb is measured in watts, and it is calculated as the energy expended in Joules per second. Strictly speaking W_u is not expended by Atwood's machine (AM), but is transformed from gravitational potential energy associated with M_h into gravitational energy associated with M_l.

For that reason I have labelled this version of power as P_s, in which the 's' stands for 'stored'.

But, we can also define power as the energy expended per second.

$P_e = W_e/T$ where the 'e' stands for 'expended'.

$$P_e = \frac{DgM_d}{\left(2DM_t/gM_d\right)} = \left(\frac{Dg^3{M_d}^3}{2M_t}\right)^{1/2}$$	Equ 9

The perspective in which energy is consumed and expelled would seem to me to be the more standard definition of power. However, both are measures of energy per second, whether it is energy transformed and stored per second, or energy expended and expelled to a sink per second. So, both are valid concepts of 'power'.

9 Step F – Make a scatter graph of Atwood's Machine's Power (P_{am}) vs Efficiency.

Neither equation 8 nor 9 can be plotted directly using Excel since they are not functions of a single variable, nor are the variables independent of each other. I therefore use a scatter graph. This means I can calculate E for sets of inputs, and P_s (or P_e) for the same variables, and plot the points that result on a P vs E graph. For reference purposes, I am calling the curve so produced the P_{am} curve, meaning the power of Atwood's Machine. So, using the above formulae (See the Ref 4 spreadsheet), I tried three things:

- I set M_h to 10 and ranged M_l from 0 to 10. (See Figure 2.)
- I set M_l to 10 and ranged M_h from 10 to 20. (See Figure 3.)
- I set M_t constant at 20 and incrementally changed M_l and M_h such that M_h goes from 10 to 20 while M_l goes from 10 to 0. (See Figure 4.)

Figure 2 – Power vs Efficiency in Atwood's Machine

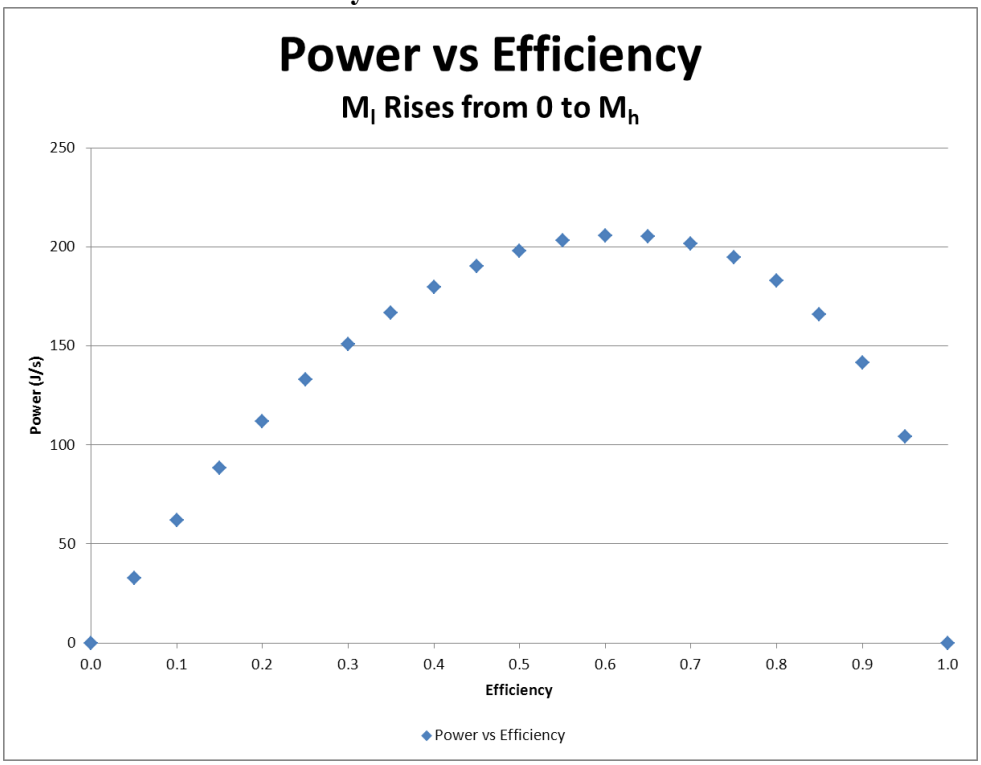


Figure 3 – Power vs Efficiency in Atwood's Machine

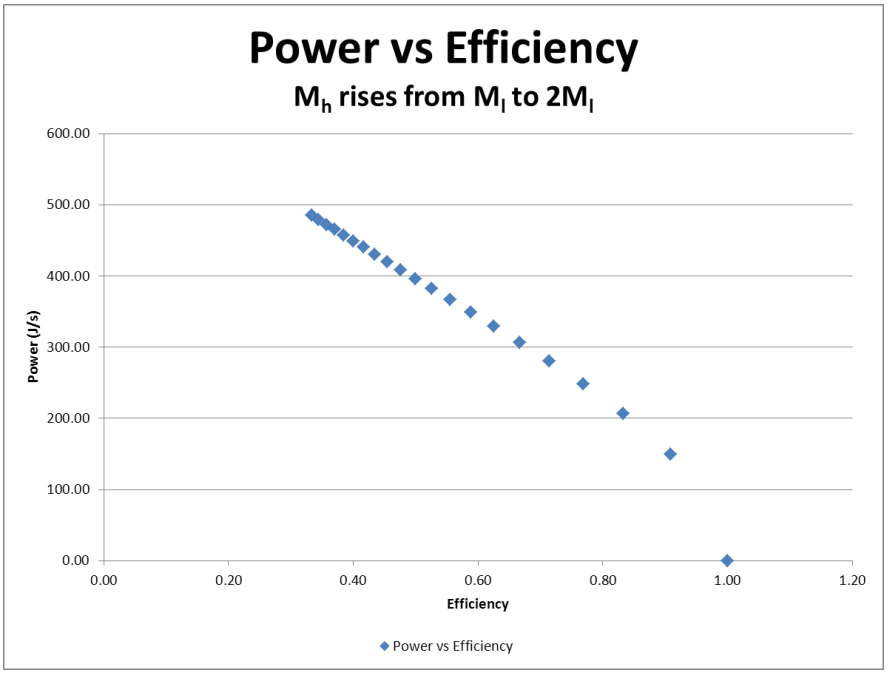


Figure 4 seems to be the graph I am looking for. It looks like the one in the Ref 2 article written by Robert Costanza. But, this shape in Figure 3 also seems very familiar to me. I could see that as soon as I read the Ref 2 article, and was confirmed in my opinion when I tried to put a quadratic trend line through it. This looks remarkably similar in shape to the envelope (the bounding curve or 'E-curve') I saw for the space of entropic indices derived from states in the state space of an EiLab system. (See figure 5 below.) Note how the quadratic trend line is too pointy at the top, and too wide at the bottom, in both graphs.

Figure 4 is a scatter graph. It does not require knowledge of the formula for the function being plotted in terms of the single variable on the horizontal axis. It only requires that P_s and E can each be calculated, for some set of variables, and then together plotted as a point.

This Figure 5 graph is 'entropic index' versus the total wealth in an **extremely** simple agent-based model economy in which the number of agents and total cash are conserved. This figure 4 graph is pulled from the Ref 6 draft paper. In this capital exchange economy (see Ref 5 for a relatively complete description) wealth is constrained to \$1, \$2, \$3 or \$4, and there are 8 agents. Since wealth is conserved, the system is endowed with an amount of cash at the beginning, and that determines the possible means of distribution. So the minimum total wealth is \$8 (each agent gets \$1, and no transactions are allowed), and the maximum total wealth is \$32 (each agent gets \$4, and no transactions are allowed). No transactions are allowed because there is only one way to allocate the wealth in those two extreme cases. For any initial endowment strictly between \$9 and \$31 more options are opened. For example, an endowment of \$10 can be distributed as 7 agents get \$1 and 1 agent gets \$3, or it can be 6 agents get \$1 and 2 agents get \$2 each. There are two distinct ways to distribute the \$10, two distinct value of entropic index computable, and two 'points' plotted on the graph for the \$10 total wealth level.

But, when the initial endowment of total wealth is exactly \$20, then there are 13 ways to allocate the wealth, and there are 13 distinguishable states in the entire state space of the system, and seven distinct values of entropic index possible. See the middle column of points that reach all the way up to entropic index of 1.00. The 13 states are represented by the thirteen points in the centre of the graph, but 12 occur in overlapping positions, so only seven are visible in the graph. Each vertical column of states (points or dots) represents the entire state space of the closed system for a given initial endowment of total wealth. The circled dots represent the maximum entropic index possible for the associated total wealth in its closed system. When the model is run, the entropic index does indeed tend to rise to the indicated maxima, perturbed by severe fluctuations.

The maxima do not form a quadratic, as can be seen by the least squares quadratic fit (for which the equation is provided). The points are the set of all states for all possible initial endowments for this system. There is a draft paper in which I develop an expression for the bounding curve which encloses all of these points (equation 22 at Ref 7, excerpted and included below). That curve looks to be very similar to the curve that comes out of the AM. (Compare figures 1, 4 and 5.)

10 Discussion

I wonder if there is an equation that expresses P_s (power of storage) in terms of E (efficiency). Now, that would be cool! Then I could compare it with my equation for the enclosing curve (E-Curve) of figure 5 (To be clear, the E-curve is not shown in figure 5, but can be imagined as the bounding curve passing just above the maximal values shown as circled dots.)

What is the formula for P_s in terms of E and constants? Can this be done? In other words, can this be transformed into the form $P_s = c * f(E)$ on the interval $0 <= E <= 1$, where c is a constant. The constant c would be some calculated value depending on the constants of the problem. In this case M_t, D, and g.

So, my goal is to produce something like this:

$P_s = c * f(E)$	Equ 10

Here are some equations to work from:

$P_s = {W_u}/{T}$	Equ 11
$E = {W_u}/{(W_u + W_e)} => W_u = \frac{E}{(1-E)} W_e$	Equ 12

Substitute W_u from equation 12, W_e from equation 4, and T from equation 2 into equation 11.

$P_s = \frac{E}{(1-E)} \times DgM_d \times \left(\frac{gM_d}{2DM_t}\right)^{1/2}$	Equ 13

Squaring temporarily to avoid the root:

$P_s{}^2 = \left(\frac{E}{(1-E)}\right)^2 \times D^2 g^2 M_d{}^2 \times \frac{gM_d}{2DM_t}$	Equ 14

Pulling out constants:

$P_s{}^2 = \left(\frac{D^2 g^3}{2DM_t}\right)\left(\frac{E}{(1-E)}\right)^2 \times M_d{}^3$	Equ 15

Putting together $M_t=M_l+M_h$; $M_d=M_h-M_l$; and $E=M_l/M_h$ we get:

$M_d = M_t \frac{(1-E)}{(1+E)}$	Equ 16

Subbing equation 16 into 15 produces:

$P_s{}^2 = \left(\frac{D g^3 M_t{}^2}{2}\right) \frac{E^2(1-E)}{(1+E)^3}$	Equ 19

Take the square root, and voila:

11 Equation for P_e

Using a technique similar to that above I can develop a formula for P_e = c g(E) as follows:

Start with the definition of P_e:

$$P_e \equiv {E_k}/{T} \equiv {W_e}/{T}$$	Equ 21

By a similar train of logic that led to equation 15 I get:

$$P_e{}^2 = \left(\frac{D^2 g^3}{2DM_t}\right)\left(\frac{E}{(1-E)}\right)^2 \times M_l{}^2 M_d$$	Equ 22

But $M_l{}^2$ M_d resolves to:

$$M_l{}^2 M_d = M_t{}^3 \frac{(1-E)}{(1+E)^3}$$	Equ 23

So, I am a little bothered by this equation 23. When I compare it to equation 16 there is a notable difference. Equation 16 has an equal power of E top and bottom. When M_d is cubed, the top and bottom are both cubic in E. On the other hand, in this expression the top is linear in E and the bottom is cubic in E. This strikes me as odd, but I see no error.

Ah! Wait! Now I see it. The E on the bottom causes the power to grow to infinity as efficiency approaches zero.

Pushing on, I get this expression for P_e:

$$P_e = \text{c g(E)} = \sqrt[2]{\frac{Dg^3 M_t{}^2}{2}} \times \sqrt[2]{\frac{(1-E)^3}{E^2(1+E)^3}}$$	Equ 24

Putting this into a spreadsheet, I get the Figure 7 graph.

15 Unresolved Issues

- What does all this mean?

16 Material Added 150101

After completing the above, I was concerned that, in using the formula mgd for potential energy I might be overlooking an important part because g is an constant and so-called 'negligible' bits are discarded.

The equation for calculating the complete gravitational potential of a three-body system is:

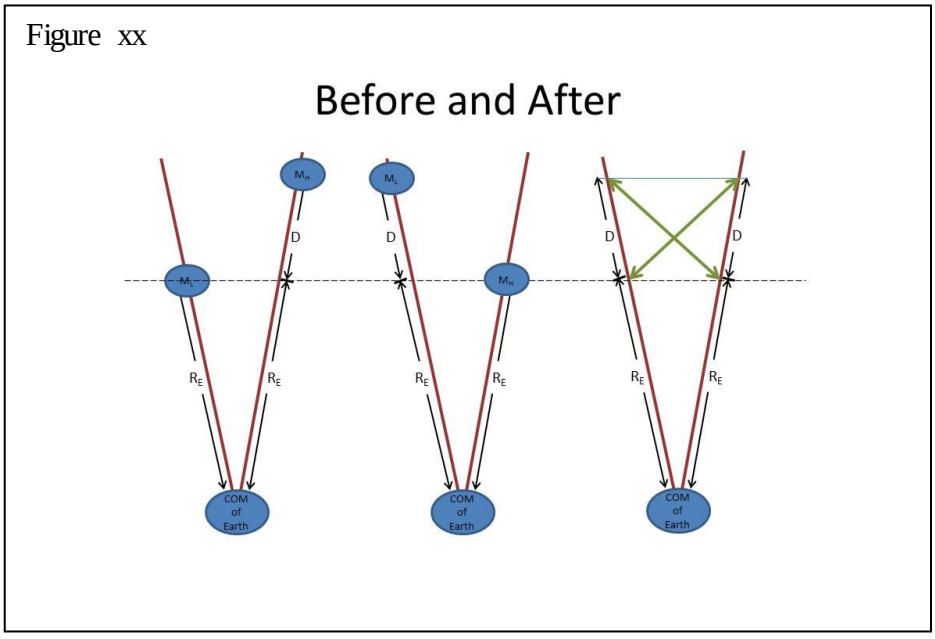
$$E = -G\sum_{i<j}\frac{m_i m_j}{\lvert\vec{r_i}-\vec{r_j}\rvert}$$	Equ 34

Where the position vectors r are from the centre of mass of the system to the centre of mass of each mass in the system. The difference of two such position vectors is a vector from the centre of mass of one mass to the centre of mass of the other. If we let the mass of the Earth be M_E then this becomes and apply this to the AM we get:

$$E = -G\left[\frac{M_L M_E}{\lvert\overrightarrow{r_{LE}}\rvert}+\frac{M_H M_E}{\lvert\overrightarrow{r_{HE}}\rvert}+\frac{M_L M_H}{\lvert\overrightarrow{r_{LH}}\rvert}\right]$$	Equ 35

But I am interested in the change in potential energy. Let RE be the radius of the Earth. Before the AM runs $r_{LE} = R_E$ and $r_{HE} = R_E+d$. After, $r_{LE} = R_E + D$ and $r_{HE} = R_E$. ΔE is then $E_{after} - E_{before}$. And, I can tell from the diagram (Figure xx) that r_{LH} is the same before and after.

Figure xx



$\Delta E = -G\left[\left(\frac{M_L M_E}{R_E + D} + \frac{M_H M_E}{R_E} + \frac{M_L M_H}{r_{LH}}\right) - \left(\frac{M_L M_E}{R_E} + \frac{M_H M_E}{R_E + D} + \frac{M_L M_H}{r_{LH}}\right)\right]$	Equ 36

The MLMH terms cancel out exactly. No approximations there. I was hoping that would happen. The other terms can be simplified by first giving them a common denominator:

$\Delta E = -G\left[\frac{R_E M_L M_E}{R_E(R_E + D)} + \frac{(R_E + D) M_H M_E}{R_E(R_E + D)} - \frac{(R_E + D) M_L M_E}{R_E(R_E + D)} - \frac{R_E M_H M_E}{R_E(R_E + D)}\right]$	Equ 37

Pulling out the common denominator and the common factor M_E:

$\Delta E = -\frac{G M_E}{R_E(R_E + D)}[R_E M_L + (R_E + D) M_H - (R_E + D) M_L - R_E M_H]$	Equ 38

Then, focusing on the terms with R_E, four terms cancel, leaving two:

$\Delta E = -\frac{G M_E}{R_E(R_E + D)}[D M_H - D M_L]$	Equ 39

Which becomes:

$\Delta E = -\frac{G M_E}{R_E(R_E + D)} D M_D$	Equ 40

Compare that with the result using MgD for potential energy:

$\Delta E = -g D M_D \; where \; \frac{G M_E}{R_E(R_E + D)} \cong g$	Equ 41

Hmm! $G \cong 6.673 \times 10^{-11}$, $M_E \cong 5.983 \times 10^{24}$ kg, and $R_E \cong 6.368 \times 10^{6}$ m. Then g would be $\cong$ 9.847 m/s^2. The D in Atwood's Machine would need to be kilometers high in order to affect the analysis in the third digit after the decimal.